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Plasma boundary as a Mach surface

P. C. STANGEBY and J. E. ALLEN
Department of Engineering Science and University College, Oxford
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Abstract. Ion motion in a plasma is treated by the equations of compressible fluid flow. The plasma–sheath boundary is identified as a closed Mach surface thus demonstrating that the Bohm criterion applies to the component of ion velocity normal to the sheath edge.

1. Introduction

For ion motion normal to the plasma–sheath boundary, Bohm (1949) has established that the ions must be accelerated by the plasma fields to a speed of $(kT_e/M_i)^{1/2}$ before reaching the sheath, where T_e is the electron temperature, M_i the ion mass and k is Boltzmann's constant. In the case of ion motion with a tangential velocity component, it will be shown below that the generalized Bohm criterion requires that the speed in the direction normal to the plasma–sheath boundary attain the value $(kT_e/M_i)^{1/2}$ independent of the tangential velocity. Such tangential ion motion is found, for example, when an electrostatic probe is placed in a plasma with an anisotropic ion velocity distribution such as the positive column of a low-pressure discharge. The model employed assumes cold ions, $T_i = 0$, and exploits the mathematical equivalence of the resulting equations and the equations of compressible potential fluid flow.

2. Basic equations

The ion motion in the plasma is described by the equations of fluid flow. The collisionless momentum equation for inviscid, zero-temperature flow is

$$M_i \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = e\boldsymbol{E} \tag{1}$$

where $\boldsymbol{q}$ is the velocity of the fluid and $\boldsymbol{E}$ is the electric field. Equation (1) can be rewritten

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla(\tfrac{1}{2}q^2) - \boldsymbol{q} \times \boldsymbol{w} = \frac{e\boldsymbol{E}}{M_i} \tag{1a}$$

where $\boldsymbol{w} = \nabla \times \boldsymbol{q}$ is the rotation of the fluid. Irrotationality of the flow, $\boldsymbol{w} = 0$, is established from equation (1) which, using $\boldsymbol{E} = -\nabla V$, may be rewritten

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = -\frac{e}{M_i}\nabla V. \tag{2}$$

Let A be an arbitrary closed curve moving with the fluid, then

$$\oint_A \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{l} = \int_A \mathrm{curl}\left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right) \cdot \mathrm{d}\boldsymbol{A} = 0 \tag{3}$$

since curl grad $V = 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \boldsymbol{q} \,.\, \mathrm{d}\boldsymbol{l} = 0$$

or $\oint \boldsymbol{q} \,.\, \mathrm{d}\boldsymbol{l}$ = constant (Kelvin's theorem), so that, if at one time or position the flow is irrotational, it cannot subsequently acquire rotation. We shall assume that the flow originates from a region of constant $\boldsymbol{q}$ hence $\boldsymbol{w} = 0$ everywhere.

Should the flow be two-dimensional or axially symmetric then no assumption about the flow at infinity is necessary to establish irrotationality, for considering conservation of energy in the steady state:

$$\tfrac{1}{2}M_i q^2 + eV = \text{constant}$$

we have

$$\nabla(\tfrac{1}{2}q^2) + \frac{e}{M_i}\nabla V = 0. \tag{4}$$

Comparing equations (4) and (1*a*) one obtains

$$\boldsymbol{q} \times \boldsymbol{w} = 0.$$

If the flow is two-dimensional or axially symmetric then $\boldsymbol{q}$ and $\nabla \times \boldsymbol{q}$ are mutually perpendicular, hence $\boldsymbol{w} = 0$ for finite $\boldsymbol{q}$. Since $\nabla \times \boldsymbol{q} = 0$ we may introduce the velocity potential ϕ such that

$$\boldsymbol{q} = -\nabla\phi. \tag{5}$$

The continuity equation for the ions is

$$\frac{\partial N}{\partial t} + \nabla \,.\, (N\boldsymbol{q}) = 0 \tag{6}$$

where N is the ion (or electron) density.

In the plasma the Boltzmann relation is assumed:

$$N_e = N_i = N = N_0 \exp\left(\frac{eV}{kT_e}\right) \tag{7}$$

where $N = N_0$ when $V = 0$.

For the steady state the combination of equations (1*a*), (5), (6) and (7) yields

$$\nabla^2\phi = \frac{1}{2a^2}\nabla\phi \,.\, \nabla(\nabla\phi \,.\, \nabla\phi) \tag{8}$$

where $a = (kT_e/M_i)^{1/2}$—the Bohm speed.

3. Plasma–sheath boundary as a closed Mach surface

Equation (8) may be recognized as being mathematically equivalent to the fundamental equation for compressible potential fluid flow with a interpreted as a sound speed, and thus all mathematical results of the latter theory may be appropriated to describe the ion fluid motion. In particular we wish to utilize the concept of the *Mach surface*. In a region of supersonic flow, $q > a$, a Mach surface is such that the fluid velocity component normal to the surface is equal to the sound speed a. We wish to identify a plasma–sheath boundary as a closed Mach surface. To do so it must be shown that:

(i) A plasma–sheath boundary can only occur on a closed Mach surface, i.e. a necessary condition for sheath formation is that the normal velocity component be a.

(ii) If a closed Mach surface exists the plasma solution blows up on it, that is a sheath forms at this surface, i.e. a sufficient condition for sheath formation is that the normal velocity component reaches a.

4. Necessary condition

This condition is established directly from compressible fluid flow theory. At a plasma–sheath boundary the *plasma solution* can no longer be extended using step-by-step integration of the differential equations, i.e. the plasma and sheath solutions are analytically different. From Courant and Friedrichs (1948, p. 55) "Whenever the flow in two adjacent regions is described by expressions which are analytically different then the two regions are necessarily separated by a characteristic" (Mach surface).

5. Sufficient condition

Consider any simple contour C in two-dimensional configuration space (two dimensions will be considered for simplicity), and the variable pair (n, s) where s is measured along C and n normally to C. We wish to show that if $C = C_M$ a closed Mach line, then $\partial q_n/\partial n \rightarrow \infty$ everywhere on C_M, i.e. the plasma solution blows up on C_M.

The differential equations are written in terms of (n, s) by considering figure 1. Combining the momentum equation and the Boltzmann relation gives

$$q_n \frac{\partial q_n}{\partial n} + q_s \frac{\partial q_s}{\partial n} = -\frac{a^2}{N} \frac{\partial N}{\partial n} \tag{9}$$

$$q_n \frac{\partial q_n}{\partial s} + q_s \frac{\partial q_s}{\partial s} = -\frac{a^2}{N} \frac{\partial N}{\partial s}. \tag{10}$$

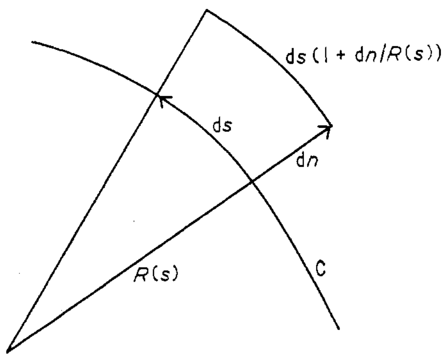

Figure 1. Diagram defining n, s and $R(s)$

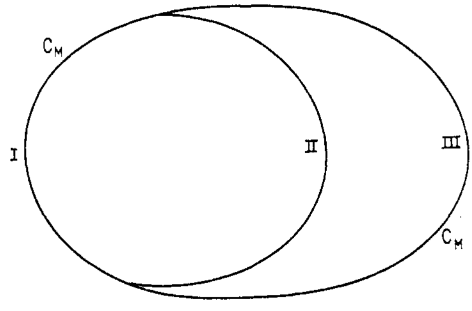

Figure 2. Diagram to show that $\partial q_n/\partial n \rightarrow \infty$ on all of C_M.

The time-independent continuity equation becomes

$$\frac{\partial(Nq_n)}{\partial n} + \frac{\partial(Nq_s)}{\partial s} + \frac{Nq_n}{R(s)} = 0 \tag{11}$$

where $R(s)$ is the radius of curvature of C at s. Equation (11) is obtained by considering the net flux out of the elemental region shown in figure 1. The irrotationality condition, $\nabla \times \boldsymbol{q} = 0$, becomes

$$\frac{\partial q_n}{\partial s} - \frac{\partial q_s}{\partial n} - \frac{q_s}{R(s)} = 0. \qquad (12)$$

Combining equations (9) to (12):

$$\frac{\partial q_n}{\partial n}\left(1 - \frac{q_n^2}{a^2}\right) + \frac{q_n}{R(s)}\left(1 + \frac{q_s^2}{a^2}\right) + \frac{\partial q_s}{\partial s}\left(1 - \frac{q_s^2}{a^2}\right) - \frac{2q_n}{a^2}\frac{\partial q_n}{\partial s} = 0. \qquad (13)$$

Now if C = C_M then $q_n = -a$ and $\partial q_n/\partial s = 0$, hence

$$\lim_{q_n \to -a} \left\{\frac{(1-q_n^2/a^2)}{(1+q_s^2/a^2)}\frac{\partial q_n}{\partial n}\right\} - \frac{a}{R(s)} + \frac{(1-q_s^2/a^2)}{(1+q_s^2/a^2)}\frac{\partial q_s}{\partial s} = 0. \qquad (14)$$

Multiply equation (13) by ds and integrate around C_M. The following two properties of contour integration may be noted:

(i) $\oint \{1/R(s)\}\mathrm{d}s = 2\pi$ for any closed contour,

(ii) $\oint f(y)\,(\partial y/\partial s)\,\mathrm{d}s = 0$ for any closed contour and y a physical variable defined on the contour.

(Strictly these results are valid for simple closed Jordan curves.) Thus one obtains

$$\lim_{q_n \to -a} \oint_{C_M} \frac{(1-q_n^2/a^2)}{(1+q_s^2/a^2)}\frac{\partial q_n}{\partial n}\mathrm{d}s = 2\pi a$$

hence $\partial q_n/\partial n \to \infty$ on some portion at least of C_M.

It may be shown that $\partial q_n/\partial n \to \infty$ on all of C_M as follows. Consider C_M as shown in figure 2, where on portion I $\partial q_n/\partial n \to \infty$ but does not do so on portion III. Now there must exist a curve such as portion II on which $\partial q_n/\partial n \to \infty$ in order to close the sheath formed by portion I. Since portion II is a sheath then $q_n = -a$ here as well (from the necessity condition).

Set

$$f \equiv \lim_{q_n \to -a} \frac{1}{a}\frac{(1-q_n^2/a^2)}{(1+q_s^2/a^2)}\frac{\partial q_n}{\partial n}$$

then

$$\int_{\text{I}} f\,\mathrm{d}s + \int_{\text{II}} f\,\mathrm{d}s - \int_{\text{I}} \frac{\mathrm{d}s}{R} - \int_{\text{II}} \frac{\mathrm{d}s}{R} = 0. \qquad (15)$$

$$\int_{\text{I}} f\,\mathrm{d}s + \int_{\text{III}} f\,\mathrm{d}s - \int_{\text{I}} \frac{\mathrm{d}s}{R} - \int_{\text{III}} \frac{\mathrm{d}s}{R} = 0. \qquad (16)$$

Subtracting equation (14) from (13)

$$\int_{\text{II}} f\,\mathrm{d}s - \int_{\text{III}} f\,\mathrm{d}s - \int_{\text{II}} \frac{\mathrm{d}s}{R} + \int_{\text{III}} \frac{\mathrm{d}s}{R} = 0.$$

Now

$$\int_{\text{I}} \frac{\mathrm{d}s}{R} + \int_{\text{II}} \frac{\mathrm{d}s}{R} = \int_{\text{I}} \frac{\mathrm{d}s}{R} + \int_{\text{III}} \frac{\mathrm{d}s}{R} = 2\pi$$

thus

$$\int_{\text{II}} \frac{\mathrm{d}s}{R} = \int_{\text{III}} \frac{\mathrm{d}s}{R}$$

and

$$\int_{\text{II}} f \, ds = \int_{\text{III}} f \, ds.$$

However, on portion III

$$\frac{\partial q_n}{\partial n} \nrightarrow \infty$$

hence

$$\int_{\text{II}} f \, ds = 0$$

and, since f is the same sign everywhere on C_M, then on portion II $\partial q_n/\partial n \nrightarrow \infty$ in contradiction to the above. If portion I expands to occupy all of C_M then this contradiction is avoided.

The possibility of multiple segments where $\partial q_n/\partial n \nrightarrow \infty$ is treated similarly, each segment being considered in turn.

6. Conclusions

It has been demonstrated that a sufficient and necessary condition for sheath formation is that the plasma fields should accelerate the ions until their velocity normal to the sheath is equal to the Bohm speed, independent of the ion velocity component tangential to the sheath.

References

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